

Counterexample to “Sufficient conditions for uniqueness of the Weak Value” by J. Dressel and A. N. Jordan, arXiv:1106.1871v1.

Stephen Parrott*

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Abstract

The abstract of “Contextual Values of Observables in Quantum Measurements” by J. Dressel, S. Agarwal, and A. N. Jordan [Phys. Rev. Lett. **104** 240401 (2010)] (called DAJ below), states:

“We introduce contextual values as a generalization of the eigenvalues of an observable that takes into account both the system observable and a general measurement procedure. This technique leads to a natural definition of a general conditioned average that converges uniquely to the quantum weak value in the minimal disturbance limit.”

A counterexample to the claim of the last sentence was presented in [4], a 32-page paper discussing various topics related to DAJ, and a simpler counterexample in Version 1 of the present work. Subsequently Dressel and Jordan placed in the arXiv the paper of the title (called DJ below) which attempts to prove the claim of DAJ quoted above under stronger hypotheses than given in DAJ, hypotheses which the counterexample does not satisfy. The present work (Version 5) presents a new counterexample to this claim of DJ.

A brief introduction to “contextual values” is included. Also included is a critical analysis of DJ.

1 Introduction

A counterexample to a major claim of

J. Dressel, S Agarwal, and A. N. Jordan, “Contextual values of observables in quantum measurements”, Phys. Rev. Lett. **104** 240401 (2010)

*For contact information, go to <http://www.math.umb.edu/~sp>

(henceforth called DAJ) was given in [4], a 32-page paper discussing DAJ in detail. The claim in question is stated as follows in DAJ’s abstract:

“We introduce contextual values as a generalization of the eigenvalues of an observable that takes into account both the system observable and a general measurement procedure. This technique leads to a natural definition of a general conditioned average that converges uniquely to the quantum weak value in the minimal disturbance limit.”

This wording (particularly, “minimal disturbance limit”) is potentially misleading, as will be explained briefly below, and is discussed more fully in [4].

Version 1 presented a simple counterexample to the claim of the above quote based on my interpretation of the vague presentation of DAJ. A later paper by Dressel and Jordan, “Sufficient conditions for uniqueness of the Weak Value” [2] (henceforth called DJ) adjoined new (and very strong) hypotheses to DAJ which the counterexample did not satisfy and claimed to prove that the above quote was correct under these new hypotheses.

The present work presents a new counterexample to that claim. It also includes the introduction to the main ideas of DAJ of Version 1 and a critical analysis of DJ.

2 Notation and brief reprise of DAJ

To establish notation, we briefly summarize the main ideas of DAJ. The notation generally follows DAJ except that DAJ denotes operators by both boldface and circumflex, e.g., $\hat{\mathbf{M}}$, but we omit the boldface and “hat” decorations. Also, we use P_f to denote the operator of projection onto the subspace spanned by a vector f . (DAJ uses $\hat{\mathbf{E}}_f^{(2)}$.)

When we quote directly an equation of DAJ, we use DAJ’s equation number, which ranges from (1) to (10), and also DAJ’s original notation. Other equations will bear numbers beginning with (100).

Suppose we are given a set $\{M_j\}$ of measurement operators, where j is an index ranging over a finite set. We assume that the reader is familiar with the theory of measurement operators, as given, for example, in the book [5] of Nielsen and Chuang. By definition, measurement operators satisfy

$$\sum_j M_j^\dagger M_j = I \quad , \quad (100)$$

where I denotes the identity operator. With such measurement operators is associated the *positive operator valued measure* (POVM) $\{E_j\}$ with $E_j := M_j^\dagger M_j$. When the system is in a (generally mixed) normalized state ρ (represented as a positive operator of trace 1), the probability of a measurement yielding result j is $\text{Tr}[M_j^\dagger M_j \rho] = \text{Tr}[E_j \rho]$. Moreover, after the measurement, the system will

be in (unnormalized) state $M_j \rho M_j^\dagger$, which when normalized is:

$$\text{normalized post-measurement state} = \frac{M_j \rho M_j^\dagger}{\text{Tr} [M_j \rho M_j^\dagger]} \quad . \quad (101)$$

For notational simplicity, we normalize states only in calculations where the normalization factor is material.

We also assume given an operator A , representing what DAJ calls “the system observable” in the above quote. We ask if it is possible to choose real numbers α_j , which DAJ calls *contextual values*, such that

$$A = \sum_j \alpha_j E_j \quad . \quad (102)$$

This will not always be possible, but we consider only cases for which it is. When it is possible, it follows that the expectation $\text{Tr} [A\rho]$ of A in the state ρ equals the expectation calculated from the probabilities $\text{Tr} [E_j \rho]$ obtained from the POVM $\{E_j\}$, with the numerical value α_j associated with outcome j :

$$\text{Tr} [A\rho] = \sum_j \alpha_j \text{Tr} [E_j \rho] \quad . \quad (103)$$

The book [6] of Wiseman and Milburn defines a measurement to be “minimally disturbing” if the measurement operators M_j are all positive (which implies that they are Hermitian).¹ DAJ uses a slightly more general definition to define their “minimal disturbance limit” of the above quote. We shall use the definition of Wiseman and Milburn [6] because it is simpler and sufficient for our counterexample. A counterexample under the definition of Wiseman and Milburn will also be a counterexample under any more inclusive definition, such as that of DAJ.

A particularly simple kind of measurement is one in which there are only two measurement operators, P_f and $I - P_f$. Intuitively, this “measurement” asks whether the (unnormalized) post-measurement state is P_f or not. Here we are using the notation of mixed states. Phrased in terms of pure states, and assuming that the pre-measurement state ρ is pure, the measurement determines if the post-measurement state is the pure state f or a pure state orthogonal to f .

Suppose that we make a measurement with the original measurement operators M_j and then make a second measurement with measurement operators $P_f, I - P_f$. In this situation, the second measurement is called a “postselection”, and when it yields state P_f , one says that the postselection has been “successful”.

¹This is a technical definition which can be misleading if one does not realize that normal associations of the English phrase “minimally disturbing” are not implied. Further discussion can be found in [6] and [4].

Such a compound measurement may be equivalently considered as a single measurement with measurement operators $\{P_f M_j, (I - P_f) M_j\}$. “Successful” postselection leaves the system in normalized state

$$\frac{(P_f M_j) \rho (P_f M_j)^\dagger}{\text{Tr} [(P_f M_j) \rho (P_f M_j)^\dagger]} \quad , \quad (104)$$

which is pure state f (P_f in mixed state notation). This result will occur with probability $p(j, f) = \text{Tr} [(P_f M_j)^\dagger P_f M_j \rho] = \text{Tr} [M_j^\dagger P_f M_j \rho]$.

The probability $p(j|f)$ of first measurement result j given that the postselection was successful is:

$$p(j|f) = \frac{p(j, f)}{\sum_i p(i, f)} = \frac{\text{Tr} [M_j^\dagger P_f M_j \rho]}{\sum_i \text{Tr} [M_i^\dagger P_f M_i \rho]} \quad . \quad (105)$$

Hence, if we assign numerical value α_j to result j as above, the conditional expectation of the measurement *given* successful postselection is:

$${}_f \langle A \rangle := \frac{\sum_j \alpha_j \text{Tr} [M_j^\dagger P_f M_j \rho]}{\sum_i \text{Tr} [M_i^\dagger P_f M_i \rho]} \quad . \quad (106)$$

This is DAJ’s “general conditioned average”. Written in DAJ’s original notation, this reads

$${}_f \langle A \rangle = \sum_j \alpha_j^{(1)} P_{j|f} = \frac{\sum_j \alpha_j^{(1)} \text{Tr} [\hat{\mathbf{E}}_{jf}^{(1,2)} \hat{\rho}]}{\sum_j \text{Tr} [\hat{\mathbf{E}}_{jf}^{(1,2)} \hat{\rho}]} \quad . \quad (6)$$

DAJ’s theory of contextual values was motivated by a theory of “weak measurements” initiated by Aharonov, Albert, and Vaidman [10] in 1988. Intuitively, a “weak” measurement is one which negligibly disturbs the state of the system. This can be formalized by introducing a “weak measurement” parameter g on which the measurement operators $M_j = M_j(g)$ depend, and requiring that

$$\lim_{g \rightarrow 0} \frac{M_j(g) \rho M_j^\dagger(g)}{\text{Tr} [M_j(g) \rho M_j^\dagger(g)]} = \rho \quad \text{for all } \rho \text{ and } j \quad , \quad (107)$$

This says that for small g , the post-measurement state is almost the same as the pre-measurement state ρ (cf. equation (104)). We shall refer to this as “weak measurement” or a “weak limit”.

The “minimal disturbance limit” mentioned in the above quote from DAJ’s abstract presumably refers to (107) combined with their generalization of Wiseman and Milburn’s “minimally disturbing” condition that the measurement operators be positive, and this is the definition that we shall use.²

²DAJ only partially and unclearly defines its “minimally disturbing” condition, but in a message to Physical Review Letters (PRL) in response to a “Comment” paper that I sub-

DAJ claims that in their “minimal disturbance limit” (which is implied by a weak limit with positive measurement operators), their “general conditioned average” ${}_f\langle A \rangle$ (6), our (106), is always given by:

$${}_f\langle A \rangle = \frac{1/2\text{Tr} [P_f\{A, \rho\}]}{\text{Tr} [P_f\rho]} \quad . \quad (108)$$

Our equation (108) is equation (7) of DAJ:

$$A_w = \frac{\text{Tr} [\hat{E}_f^{(2)}\{\hat{A}, \hat{\rho}\}]}{2\text{Tr} [\hat{E}_f^{(2)}\hat{\rho}]} \quad , \quad (7)$$

Here A_w is their notation for “weak value” of A .³

The statement of DAJ quoted in the Introduction, that their

“... general conditioned average ... converges uniquely to the quantum weak value in the minimal disturbance limit”,

implies that for a weak limit of positive measurement operators, their (6) always evaluates to (7), or in our notation, our (106) always evaluates to (108). We shall give an example for which (106) does *not* evaluate to (108).

3 The counterexample

3.1 General discussion

We are assuming the “minimal disturbance” condition that the measurement operators be positive, so in the definition (106) of DAJ’s “general conditioned average”, we replace M_j^\dagger with M_j . First we examine its denominator.

Let

$$\eta_j(g) := \text{Tr} [M_j(g)\rho M_j(g)] \quad , \quad (109)$$

which are inverse normalization factors for the unnormalized post-measurement states $M_i(g)\rho M_i(g)$ (cf. (101). We shall assume that all $\eta_j(g)$ are bounded for

mitted, the authors of DAJ confirmed that Wiseman and Milburn’s definition implies theirs. DAJ uses but does not define the phrase “weak limit”, but in the same message to PRL, the authors state that (107) corresponds to “ideally weak measurement”. Since “ideally weak measurement” must be (assuming normal usage of syntax) a special case of mere “weak measurement”, our counterexample which assumes (107) will also be a counterexample to the statement of DAJ quoted in the introduction.

I have made several direct inquiries to the authors of DAJ requesting a precise definition of their “minimal disturbance limit”, but all have been ignored.

³In the traditional theory of “weak measurement” initiated by [10], the weak limit (i.e., $\lim_{g \rightarrow 0}$) of (106) (equivalently, (6)) would be called a “weak value” of A , though the traditional “weak measurement” literature calculates it via different procedures. When ρ is a pure state, most modern authors calculate this weak value as (108) (equivalently (7)), though the seminal paper [10] arrived (via questionable mathematics) at a complex weak value of which (108) is the real part. (Only recently was it recognized that “weak values” are not unique [7][8][9].)

small g , which is expected (because we expect $M_j(g)$ to approach a multiple of the identity for small g in order to make the measurement “weak”) and will be the case for our counterexample. We have

$$\begin{aligned}
\lim_{g \rightarrow 0} \sum_j \text{Tr} [P_f M_j(g) \rho M_j(g)] &= \\
\lim_{g \rightarrow 0} \sum_j \text{Tr} [P_f \left(\frac{M_j(g) \rho M_j(g)}{\eta_j(g)} - \rho \right) \eta_j(g)] &+ \lim_{g \rightarrow 0} \sum_j \text{Tr} [P_f \rho] \eta_j(g) \\
&= \lim_{g \rightarrow 0} \sum_j \text{Tr} [P_f \rho] \text{Tr} [M_j(g) \rho M_j(g)] \\
&= \text{Tr} [P_f \rho] \lim_{g \rightarrow 0} \text{Tr} \left[\sum_j M_j(g) M_j(g) \rho \right] \\
&= \text{Tr} [P_f \rho] \quad , \tag{110}
\end{aligned}$$

because $\sum M_j^2 = \sum M_j^\dagger M_j = I$ and $\text{Tr} [\rho] = 1$. This is the denominator of DAJ’s claimed result (108) (half the denominator of their (7) because both numerator and denominator of our (108) differ from (7) by a factor of 1/2).

Next we examine the numerator of the “general conditioned average” (106). We shall write it as a sum of two terms, the first term leading to DAJ’s (108), and the second a term which does not obviously vanish in the limit $g \rightarrow 0$. The counterexample will be obtained by finding a case for which the limit of the second term actually does not vanish.

Note the trivial identity for operators M, ρ :

$$M \rho M = M[\rho, M] + M^2 \rho$$

and the similar

$$M \rho M = -[\rho, M]M + \rho M^2 \quad .$$

Combining these gives

$$M \rho M = \frac{1}{2} \{M^2, \rho\} + \frac{1}{2} [M, [\rho, M]] \quad . \tag{111}$$

Using (111) and the contextual value equation (102), $A = \sum_j \alpha_j E_j = \sum_j \alpha_j M_j^2$, we can rewrite the numerator of (106) as

$$\begin{aligned}
\text{numerator of (106)} &= \sum_j \alpha_j \text{Tr} [M_j P_f M_j \rho] \\
&= \sum_j \alpha_j \text{Tr} [P_f M_j \rho M_j] \\
&= \frac{1}{2} \text{Tr} [P_f \{A, \rho\}] + \sum_j \frac{1}{2} \alpha_j \text{Tr} [P_f [M_j, [\rho, M_j]]] \quad . \tag{112}
\end{aligned}$$

After division by the denominator of (106), the first term gives DAJ's claimed (7) in the limit $g \rightarrow 0$, our (108), and the second term gives

difference between weak limit of (6) and (7) =

$$\lim_{g \rightarrow 0} \frac{\sum_j \frac{1}{2} \alpha_j(g) \text{Tr} [P_f [M_j(g), [\rho, M_j(g)]]]}{\text{Tr} [P_f \rho]}. \quad (113)$$

We shall call (113) the “anomalous term”. Since there is no obvious control over the size of the $\alpha_j(g)$, a counterexample is expected, but was surprisingly hard to find.

The Version 1 counterexample for the quoted claim of DAJ and the newer counterexample for the new claim of DJ are identical up to this point. The difference is that the Version 1 counterexample used three 2×2 diagonal matrices as measurement operators, resulting in a contextual value equation (102) with multiple solutions, whereas the newer counterexample uses three 3×3 diagonal matrices for which there is a unique solution to (102). The newer counterexample could supercede the Version 1 example, but we retain the original counterexample because of its simple and intuitive nature (e.g., all steps can be mentally verified).

3.2 The Version 1 counterexample

The “system observable” A for the counterexample will correspond to a 2×2 matrix

$$A := \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad (114)$$

There will be three measurement operators:

$$\begin{aligned} M_1(g) &:= \begin{bmatrix} 1/2 + g & 0 \\ 0 & 1/2 - g \end{bmatrix}, \quad M_2(g) := \begin{bmatrix} 1/2 - g & 0 \\ 0 & 1/2 + g \end{bmatrix} \\ M_3(g) &:= \begin{bmatrix} \sqrt{1/2 - 2g^2} & 0 \\ 0 & \sqrt{1/2 - 2g^2} \end{bmatrix}. \end{aligned} \quad (115)$$

Note that $M_3(g)$ is uniquely defined by the measurement operator equation $\sum_{j=1}^3 M_j^2(g) = 1$ and that all three measurement operators approach multiples of the identity as $g \rightarrow 0$, which assures weakness of the measurement. Note also that $M_3(g)$ is actually a multiple of the identity for all g , so the commutators in the expression (113) for the anomalous term which involve M_3 vanish. That is, M_3 , and hence α_3 , make no contribution to the anomalous term.

Writing out the contextual value equation (102) in components gives two scalar equations in three unknowns:

$$\begin{aligned} (1/2 + g)^2 \alpha_1(g) + (1/2 - g)^2 \alpha_2(g) + (1/2 - 2g^2) \alpha_3(g) &= a \\ (1/2 - g)^2 \alpha_1(g) + (1/2 + g)^2 \alpha_2(g) + (1/2 - 2g^2) \alpha_3(g) &= b \end{aligned} \quad (116)$$

The solution can be messy because of the algebraic coefficients. However, for the case $a = 1 = b$, a solution can be obtained without calculation. This choice of a and b corresponds to the system observable being the identity operator, so the measurement is not physically interesting, but it gives a mathematically valid example with minimal calculation. Later we shall indicate how counterexamples can be obtained for other choices of a and b from appropriate solutions of (116).

Assuming $a = 1 = b$, the system (116) can be rewritten

$$\begin{aligned} (1/2 + g)^2 \alpha_1(g) + (1/2 - g)^2 \alpha_2(g) &= 1 - (1/2 - 2g^2) \alpha_3(g) \\ (1/2 - g)^2 \alpha_1(g) + (1/2 + g)^2 \alpha_2(g) &= 1 - (1/2 - 2g^2) \alpha_3(g) \end{aligned} \quad (117)$$

We will think of $\alpha_3(g)$ as a free parameter to be arbitrarily chosen, and as noted previously, the choice will not affect the anomalous term (113).

Viewed in this way, (117) becomes a system of two linear equations in two unknowns which become the same equation if $\alpha_2 = \alpha_1$, with solution

$$\alpha_2(g) = \alpha_1(g) = \frac{1 - (1/2 - 2g^2) \alpha_3(g)}{(1/2 + g)^2 + (1/2 - g)^2} = \frac{1 - (1/2 - 2g^2) \alpha_3(g)}{1/2 + 2g^2}. \quad (118)$$

Since α_3 can be chosen arbitrarily, also $\alpha_2 = \alpha_1$ can be arbitrary; we shall choose $\alpha_3(g)$ so that

$$\alpha_2(g) = \alpha_1(g) = \frac{1}{g^2}. \quad (119)$$

To see that this solution will produce a counterexample, note that for

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$$

and for any diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \quad [D, \rho] = \begin{bmatrix} 0 & (d_1 - d_2) \rho_{12} \\ (d_2 - d_1) \rho_{21} & 0 \end{bmatrix}, \text{ and}$$

$$[D, [D, \rho]] = \begin{bmatrix} 0 & (d_1 - d_2)^2 \rho_{12} \\ (d_2 - d_1)^2 \rho_{21} & 0 \end{bmatrix}.$$

In particular for $j = 1, 2$,

$$[M_j(g), [M_j(g), \rho]] = \begin{bmatrix} 0 & 4g^2 \rho_{12} \\ 4g^2 \rho_{21} & 0 \end{bmatrix},$$

and since $M_3(g)$ is a multiple of the identity, $[M_3(g), \rho] = 0$. Hence (113) becomes:

$$\frac{-(1/2) \text{Tr} [P_f \sum_j \alpha_j [M_j(g), [M_j(g), \rho]]]}{\text{Tr} [P_f \rho]} = \frac{-\text{Tr} [P_f \begin{bmatrix} 0 & 4\rho_{12} \\ 4\rho_{21} & 0 \end{bmatrix}]}{\text{Tr} [P_f \rho]}. \quad (120)$$

This is easily seen to be nonzero for $\rho_{12} \neq 0$ and appropriate P_f . For a norm 1 vector $f := (f_1, f_2)$

$$\text{weak limit of (6)} = \frac{\text{Tr} [P_f \{A, \rho\}]}{2\text{Tr} [P_f \rho]} + \frac{-8\Re(f_2^* f_1 \rho_{21})}{|f_1|^2 \rho_{11} + 2\Re(f_2^* f_1 \rho_{21}) + |f_2|^2 \rho_{22}}. \quad (121)$$

The counterexample just given assumed that the system observable $A := \text{diag}\{a, b\}$ was the identity to make the calculations easy, but counterexamples can be obtained for any system observable. For example, if A is the one-dimensional projector $A := \text{diag}\{1, 0\}$, and if system (117) is solved with $\alpha_1(g) := 1/g^2$, then $\alpha_2(g) = 1/g^2 - 1/(2g)$, and the weak limit of the anomalous term is the same as just calculated for $A = I$. [4]

DJ [2] adds additional (very strong) hypotheses to those of DAJ which the counterexample just given does not satisfy.⁴ Assuming these additional conditions, DJ attempts to prove that (6) evaluates to (7) in their “minimal disturbance limit”. The next sections will present a more powerful counterexample which disproves this new claim of DJ.

Originally a new paper with the more powerful counterexample was submitted to the arXiv, but a moderator rejected it. He thought that instead, Version 1 should be modified. Rather than waste time on a lengthy and unpleasant appeal, I decided that it would be easier to do that.

The paper to this point is Version 1. The sections following comprise essentially the rejected arXiv paper, which presents the more powerful counterexample and critically analyzes DJ.

The new counterexample is fairly simple, utilizing three 3×3 matrices, but not as intuitive as one would like. It was found by analyzing the properties that measurement operators might have in order that (7) could be shown false, and then playing with parametrized 3×3 measurement operators, trying to adjust the parameters so that (7) would not hold. The Version 4 counterexample is simpler and more powerful than the Version 2 counterexample. No doubt even simpler counterexamples could be found. Besides the new counterexample, we attempt to clarify some statements in DJ which we think might be misleading.

4 The new additional hypotheses for the claim that (6) implies (7) in the “minimal disturbance limit”

Section 5 of DJ lists several additional assumptions, the most important of which are:

1. The M_j commute with each other and A (so they can all be represented by diagonal matrices).

This is a strong assumption. It is hard to imagine how it could reasonably

⁴ However, the fact that these additional conditions cannot reasonably be inferred from DAJ is not made clear by DJ, and a casual reader might well obtain the opposite impression.

be inferred or even guessed from DAJ. The closest reference in DAJ to something similar is the following.

“To *illustrate* [emphasis mine] the construction of the least redundant set of [contextual values], we consider the case when $\{\hat{M}_j\}$ and \hat{A} all commute.”

Nothing is said about this being a general assumption for the rest of the paper. Indeed, such an assumption would seriously restrict the applicability of the following definition (6) of “general conditioned average” ${}_f\langle\hat{A}\rangle$, which requires no such assumption. I studied DAJ for months without ever being led to even consider the possibility that this might be an *assumption* for the general claims of its abstract.

2. The contextual values $\vec{\alpha} = (\alpha_1, \dots, \alpha_N)$ are obtained from the eigenvalues $\vec{a} = (a_1, \dots, a_m)$ of $A = \text{diag}(a_1, \dots, a_m)$ as

$$\vec{\alpha} = F^{(+)}\vec{a}$$

where F is an $N \times m$ matrix satisfying $F\vec{\alpha} = \vec{a}$ and $F^{(+)}$ its Moore-Penrose pseudo-inverse. The Version 1 counterexample does not satisfy this condition.

Relying only on what is written in DAJ, it would be very hard for a reader to guess that this is supposed to be a *hypothesis* for (6), or for a claim that (6) implies (7), or both. (I did consider these possibilities, but rejected them as too unlikely, as will be explained later in more detail.) The only passage of DAJ which seems possibly relevant is:

“... we propose that the physically sensible choice of [contextual values] is the least redundant set uniquely related to the eigenvalues through the Moore-Penrose pseudoinverse.”

DAJ gives no reason why this should be the “physically sensible choice”. (DJ does attempt to address this issue, but unconvincingly and badly incorrectly, as will be discussed later.) Again, to assume this would seem to artificially limit the applicability of (6), since (6) is correct independently of this assumption.

We do not list the other hypotheses for DJ’s attempted proof that (6) implies (7) because they are more technical and less surprising than the two just discussed. Our counterexample will satisfy all of the hypotheses listed in DJ. The counterexample for Version 2 has been replaced by a simpler example in Version 4.

5 A counterexample to the claim of DJ

Section V of DJ entitled “General Proof” attempts to show that (6) implies (7) under their listed hypotheses. The present section presents a counterexample

which satisfies all of their listed hypotheses, yet the weak limit of their “general conditioned average” (6) is *not* the “quantum weak value” (7).

We follow identically the analysis of Section 3 through equation (113). This time, we use a system observable

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \quad (201)$$

and three measurement operators which are 3×3 diagonal matrices:

$$\begin{aligned} M_1(g) &:= \begin{bmatrix} \sqrt{1/2+g} & 0 & 0 \\ 0 & \sqrt{1/2} & 0 \\ 0 & 0 & \sqrt{1/2+g} \end{bmatrix} , \\ M_2(g) &:= \begin{bmatrix} \sqrt{1/3+g^2} & 0 & 0 \\ 0 & \sqrt{1/3+g} & 0 \\ 0 & 0 & \sqrt{1/3} \end{bmatrix} , \\ M_3(g) &:= \begin{bmatrix} \sqrt{1/6-g-g^2} & 0 & 0 \\ 0 & \sqrt{1/6-g} & 0 \\ 0 & 0 & \sqrt{1/6-g} \end{bmatrix} . \end{aligned} \quad (202)$$

The contextual values $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ satisfy $F\vec{\alpha} = \vec{a} := (1, 0, 0)^T$ with

$$F = \begin{bmatrix} 1/2+g & 1/3+g^2 & 1/6-g-g^2 \\ 1/2 & 1/3+g & 1/6-g \\ 1/2+g & 1/3 & 1/6-g \end{bmatrix} \quad (203)$$

The matrix F is invertible with inverse (which is also equal to the Moore-Penrose pseudoinverse $F^{(+)}$)

$$F^{(+)} = F^{-1} = \begin{bmatrix} \frac{1-6g}{6g^2} & \frac{1-2g}{2g} & \frac{-1+9g}{6g^2} \\ \frac{1-6g}{6g^2} & \frac{1+2g}{2g} & \frac{-1+3g}{6g^2} \\ \frac{-5-6g}{6g^2} & \frac{1+2g}{2g} & \frac{3g+5}{6g^2} \end{bmatrix} . \quad (204)$$

The important thing to note is that the first column, which is also $(\alpha_1, \alpha_2, \alpha_3)^T$, is of leading order $1/g^2$ as $g \rightarrow 0$, which is all that the subsequent proof will use:

$$\alpha_1(g) = \alpha_2(g) = \frac{1-6g}{6g^2}, \quad \alpha_3(g) = \frac{-5-6g}{6g^2}. \quad (205)$$

The full inverse (204) was obtained from a computer algebra program, and the first column (which is all that the counterexample will use) was also checked by hand using Cramer’s rule.

Equations (110) through (113) write the “general conditioned average” ${}_f\langle A \rangle$ of (6) as a sum of two terms, one of which evaluates to (7) in the weak limit

$g \rightarrow 0$. The other term, called the “anomalous term”, is given by (113) as:

difference between weak limit of (6) and (7) =

$$\lim_{g \rightarrow 0} \frac{\sum_k \frac{1}{2} \alpha_k(g) \text{Tr} [P_f [M_k(g), [\rho, M_k(g)]]]}{\text{Tr} [P_f \rho]}. \quad (206)$$

To disprove the claim of DJ, we need to show that there exists a state ρ and vector f such that the anomalous term does not vanish.

It is well-known that the only matrix S such that for all projection matrices P_f , $\text{Tr} [P_f S] = 0$, is the zero matrix $S = 0$.⁵ Hence it will be enough to show that

$$\lim_{g \rightarrow 0} \sum_k -\frac{1}{2} \alpha_k(g) [M_k(g), [M_k(g), \rho]] \neq 0. \quad (207)$$

for some mixed state ρ such that for all nonzero vectors f , $\text{Tr} [P_f \rho] \neq 0$.

First note that for any diagonal matrix $D = \text{diag}(d_1, d_2, d_3)$ and any matrix $\rho = (\rho)_{ij}$,

$$[D, [D, \rho]]_{ij} = (d_i - d_j)^2 \rho_{ij}. \quad (208)$$

In the cases of interest to us, D will be one of the measurement operators $M_k(g)$, $(d_i(g) - d_j(g))^2 = O(g^2)$ for all i, j , and for some i, j , the leading order of $(d_i(g) - d_j(g))^2$ is actually g^2 . The $\alpha_k(g)$ all diverge like $1/g^2$ as $g \rightarrow 0$. Thus we can see without calculation that we will obtain a counterexample unless some unrecognized relation forces the terms of (207) to exactly cancel.⁶

That cancellation does not occur in this case can be seen with minimal calculation as follows. In (208), take $(i, j) := (1, 2)$, and note that $(d_1 - d_2)^2$ is always non-negative. When $D = M_3(g)$, from the power series

$$\sqrt{c+x} = \sqrt{c} + \frac{x}{2\sqrt{c}} + O(x^2),$$

one sees that $(d_1 - d_2)^2 = O(g^4)$, and since $\alpha_3(g)$ is only $O(g^{-2})$, the $k = 3$ term in (207) vanishes in the limit $g \rightarrow 0$.

We also have

$$\alpha_1(g) = \alpha_2(g) = \frac{1}{6g^2} - \frac{1}{g},$$

⁵A computational proof is routine, but to see this without calculation, recall that $\langle S, T \rangle := \text{Tr} S^\dagger T$ defines a complex Hilbert space structure (i.e., positive definite complex inner product) on the set of $n \times n$ matrices. If $\langle S, T \rangle$ vanishes for all projectors $T = P_f$, then (by the spectral theorem), it vanishes for all Hermitian T , and hence for all T , in which case S is orthogonal to all elements of this Hilbert space and hence is the zero element.

⁶One useful observation that we can make from what we have done so far without detailed calculation is that the attempted proof of DJ is likely wrong or at least seriously incomplete, since that attempted proof concludes the vanishing of (207) on the basis of order of magnitude arguments only. Though framed in different language, it essentially says that (207) must vanish because they think that $\alpha_k(g) = (F^{(+)}(g)(1, 0, 0)^T)_k = O(1/g)$ (contradicting (205)), while $[M_j(g), [M_j(g), \rho]] = O(g^2)$.

and for either $D = M_1(g)$ or $D = M_2(g)$,

$$(d_1 - d_2)^2 = (g/\sqrt{2})^2 + O(g^2))^2 = g^2/2 + O(g^3)$$

So, in the limit $g \rightarrow 0$, (207) evaluates to

$$-\frac{1}{2} \frac{1}{6} \frac{1}{2} \rho_{12} = -\frac{\rho_{12}}{24} \quad . \quad (209)$$

Note that all we care about is that (209) does not always vanish, and this can be seen solely from the fact that α_1 and α_2 have the same sign, so that the $k = 1, 2$ terms in (207) are negative multiples of ρ_{12} which do not vanish identically in the limit $g \rightarrow 0$.

To finish the proof, let ρ be a positive definite state (i.e., all eigenvalues strictly positive) such that $\rho_{12} \neq 0$. Such a state can be constructed by starting with a positive definite diagonal state and adding a small perturbation to assure $\rho_{12} \neq 0$ (which will result in a positive definite state if the perturbation is small enough). Since ρ is positive definite, $\text{Tr} [\rho P_f] \neq 0$ for all nonzero vectors f , and we are done.

6 Discussion of DJ

6.1 Possible error in DJ's proof

The counterexample given above unfortunately relies on some detailed calculation. A conceptual counterexample would certainly be preferable. A reader interested in discovering the truth of the matter will be faced with the unpleasant choice of wading through DJ's dense proof or checking the boring details of the counterexample. For such readers, it may be helpful if we point out what seems a potentially erroneous step in DJ's proof.

A step which caused me to question their proof occurs at the very end of their Section V:

“... to have a pole of order higher than g^n [$n = 1$ in the counterexample] then there must be at least one relevant singular value with an order greater than g^n . [The counterexample has a singular value of order g^2 .] However, if that were the case then the expansion of F to order g^n would have a relevant singular value of zero and therefore could not satisfy (25) ...”

I have not been able to guess a meaning for “the expansion of F to order g^n would have a relevant singular value of zero” under which the last sentence would be true.

6.2 Significance of the Moore-Penrose pseudo-inverse

The original paper DAJ [1] introduced the Moore-Penrose pseudo-inverse as follows:

“... we propose that the physically sensible choice of [contextual values $\vec{\alpha}$] is the least redundant set uniquely related to the eigenvalues [$\vec{a} = (a_1, \dots, a_m)$ with $A = \text{diag}(a_1, \dots, a_m)$] through the Moore-Penrose pseudoinverse.”

I puzzled for a long time over this statement. Besides the fact that the meaning of “least redundant set” was obscure to me, they give no reason why this choice (which presumably means $\vec{\alpha} = F^{(+)}\vec{a}$, with $F^{(+)}$ the Moore-Penrose pseudo-inverse) should be considered the unique “physically sensible” choice, or even a physically sensible choice. The arXiv paper DJ which we are discussing attempts to fill this gap, but the attempt relies on erroneous mathematics and is unconvincing.

Before starting the discussion of this attempt, let me remark that although the attempt seems partly aimed at invalidating the counterexample of [3], it is basically irrelevant to that aim. That counterexample is a valid mathematical counterexample to a mathematical claim of DAJ as I imagine the vague exposition of DAJ would probably be interpreted by most readers. Though the counterexample uses a particular solution of the contextual value equation $F\vec{\alpha} = \vec{a}$, it was never claimed that this solution has any physically desirable properties. Though DJ does *not* show that the counterexample is unphysical as DJ claims, even if it were shown unphysical, it would still disprove the claim that (6) necessarily evaluates to (7) in the “minimal disturbance limit”. A reader of DAJ cannot reasonably be expected to guess that the definition of “minimal disturbance limit” is supposed to include the pseudo-inverse prescription.

Therefore, the discussion will be directed toward analyzing the claim of DJ that the pseudo-inverse solution should be preferred because DJ thinks (incorrectly) that

“... the pseudo-inverse solution will choose the solution that generally provides the most rapid statistical convergence for observable measurements on the system.”

A careful analysis of DJ’s argument for this claim will reveal flaws which invalidate it.

DJ writes:

“With the pseudo-inverse in hand, we then find a uniquely specified solution $\vec{\alpha}_0 = F^{(+)}\vec{a}$ that is directly related to the eigenvalues of the operator. Other solutions $\alpha = \vec{\alpha}_0 + x$ of (3) will contain additional components in the null space of F , and will thus deviate from this least redundant solution. [True if sympathetically interpreted, but tautological.] Consequently, the solution α_0 has the least norm of all solutions ...”

The Euclidean norm $\|\vec{\alpha}\|^2 := \sum_j \alpha_j^2$ in the real Hilbert space R^n has no physical significance in quantum theory. Why is it relevant that $\vec{\alpha}_0$ has least norm? The discussion immediately following may possibly be intended to answer this, but when analyzed it only tautologically repeats what has already been said.

However, an inattentive reader could easily get the impression that something had been proved.

DJ thinks that this immediately following discussion (at the bottom of the first column of p.4) gives “mathematical reasons for using the pseudoinverse”, but in fact no convincing reason has been given.

The next paragraph continues:

“In addition to the mathematical reasons for using the pseudoinverse in this context [referring to the discussion just analyzed, which doesn’t give any convincing mathematical reason], there is an important physical one that we will now describe. As mentioned, a fully compatible detector can be used together with the contextual values to reconstruct any moment of a compatible observable. However, since the detector outcomes are imperfectly correlated with the observable, the contextual values typically lie outside of the eigenvalue range and many repetitions of the measurement must be practically performed to obtain adequate precision for the moments. *Importantly, the uncertainty in the moments is controlled by the the variance, not of the observable operator, but of the contextual values themselves.* [emphasis mine]”

Consider a probability space with outcomes $\{1, 2, \dots, n\}$ with probability p_j for outcome j . A *random variable* v is an assignment $j \mapsto v_j$ of a real number v_j to each outcome j . The *mean* \bar{v} of v is defined as usual by

$$\bar{v} := \sum_j v_j p_j \quad ,$$

and the *variance* τ^2 of v is defined by

$$\tau^2 := \sum_j (v_j - \bar{v})^2 p_j = \sum_j v_j^2 p_j - \bar{v}^2 \quad .$$

Here we use the symbol τ^2 instead of the customary σ^2 to denote the variance to avoid confusion with the different σ^2 defined in DJ (as the second moment).

One can speak of the “variance” of a random variable on a classical probability space, or of the “variance” of quantum observable measured in a given state. But what can it mean to speak of the “variance” of contextual values α_j ? Contextual values are *predefined* to satisfy

$$A = \sum_j \alpha_j M_j^\dagger M_j \quad , \tag{102}$$

where A is the “system observable” and $\{M_j\}$ a collection of measurement operators. What is *measured* are the outcomes j .

However, even though we know the contextual values beforehand from (102), one might speak of “measuring” them in the following sense. To every outcome j corresponds a contextual value α_j . A given state of the system ρ makes the set

of all outcomes j into a probability space by assigning a probability p_j to each outcome j : $p_j = \text{Tr} [M_j^\dagger M_j \rho]$. The assignment $j \mapsto \alpha_j$ is a random variable on this probability space, and it is meaningful to speak of its “variance”. The subsequent analysis assumes that this is the meaning that DJ intended. This discussion may seem inappropriately elementary, but I was initially puzzled about this point, and it cannot hurt to make it explicit.

Note that the mean $\bar{\alpha} = \text{Tr} [A\rho]$ of this random variable is the same no matter how the contextual values α_j are chosen so long as they satisfy the contextual value equation (102). That implies that choosing the contextual values so as to minimize the true variance τ^2 in a given state is equivalent to minimizing the second moment σ^2 . Note also that the mean and variance implicitly depend on the state ρ , and that there is no reason to think that one might be able to choose the contextual values so as to minimize the variance in *all* states.

DJ continues:

“Consequently, it is in the experimenters best interests to minimize the second moment of the contextual values,

$$\sigma^2 = \sum_j \alpha_j^2 p_j, \quad (210)$$

where p_j is the probability of outcome j .”

DJ correctly identifies σ^2 as the second moment, but unless read very carefully, the subsequent discussion could encourage confusion of σ^2 with the true variance τ^2 .

Next DJ notes that $\|\vec{\alpha}\|^2$ is a (very crude) upper bound for σ^2 :

$$\sigma^2 := \sum_j \alpha_j^2 p_j \leq \sum_j \alpha_j^2 = \|\vec{\alpha}\|^2 \quad . \quad (*)$$

“In the absence of prior knowledge about the system one is dealing with, this is the most general bound one can make. Therefore, the pseudo-inverse solution will choose the solution that generally provides the most rapid statistical convergence for observable measurements on the system.”

This is highly questionable. Although it may not be clear at this point, subsequent paragraphs make clear that DJ is claiming that it is legitimate to use $\|\vec{\alpha}\|^2$ as a sort of estimate for σ^2 , the strange and invalid justification for the claim being the sentences of the quote following equation (*).

DJ’s next paragraph computes $\|\vec{\alpha}(g)\|^2$ for both the $\vec{\alpha}(g)$ used in the counterexample of [3] and for the pseudo-inverse solution $\vec{\alpha}_0(g) = F^{(+)}(g)\vec{a}$, using $\|\vec{\alpha}(g)\|^2$ as a kind of crude estimate for $\sigma^2 = \sigma^2(g) = \sigma^2(g, \rho)$.

“For the case of the counterexample, the Parrott solution (13) [(13) should be (11)] has to leading order the bound on the variance

$$\|\vec{\alpha}\|^2 = \frac{3}{g^4} - \frac{3(a-b)}{2g^3} + O\left(\frac{1}{g^2}\right), \quad (15)$$

while the pseudoinverse solution (11) [(11) should be (13)] has to leading order the bound

$$\|\vec{\alpha}\|^2 = \frac{(a-b)^2}{8g^2} + \frac{2}{3}(a+b)^2 + O(g^2). \quad (16)$$

For any observable \vec{a} , *the Parrott solution has detector variance of order $O(1/g^4)$* [emphasis mine], which would swamp any attempt to measure an observable near the weak limit. ... However, the pseudoinverse solution has a detector variance of order $O(1/g^2)$ in the worst case; ... ”

What invalidates the argument is the use of the crude upper bound (*) as an estimate for the second moment σ^2 and the subsequent claim that “the Parrott solution has detector variance of order $O(1/g^4)$... ”.⁷

Solely from *upper bounds* for two quantities, one cannot draw any reliable conclusions about the relative size of the quantities themselves. To see this clearly in a simpler context which uses essentially the same reasoning, consider the upper bounds

$$x < x^4 \quad \text{and} \quad x^2 < x^3$$

for real numbers $x > 1$. From the fact that the first upper bound x^4 (for x) is larger than the second upper bound x^3 (for x^2), we cannot conclude that x is larger than x^2 for $x > 1$. Yet DJ’s argument relies on this type of incorrect reasoning.

In the interests of following closely the exposition of DJ, we passed rapidly over (*). Let us return to analyze it more closely:

$$\sigma^2 := \sum_j \alpha_j^2 p_j \leq \sum_j \alpha_j^2 = \|\vec{\alpha}\|^2 \quad . \quad (*)$$

“In the absence of prior knowledge about the system one is dealing with, this is the most general bound one can make. Therefore, the pseudo-inverse solution will choose the solution that generally provides the most rapid statistical convergence for observable measurements on the system.”

Note once again that $\sigma^2 = \sigma^2(\rho)$ depends implicitly on the state ρ because the probabilities $p_j = \text{Tr} [\rho M_j^\dagger M_j]$ of outcome j depend on ρ . Keeping this in mind, one sees how crude the upper bound (*) really is.

For a nonzero system observable A , equality holds in (*) (i.e., $\sigma^2(\rho) = \|\vec{\alpha}\|^2$) only in the trivial case in which one particular $p_J = 1$ and the others vanish,

⁷DJ incorrectly identifies σ^2 as the “variance” instead of the second moment, but this is a mere slip. Ignoring this slip, technically one could argue that this statement is correct because to say that a quantity is $O(1/g^4)$ only means that it increases *no faster* than $1/g^4$ as $g \rightarrow 0$. For example, $g^8 = O(1/g^4)$. However, in the context and taking into account the typically sloppy use of the “big-oh” notation in the physics literature, most readers would probably interpret this passage as claiming that the “Parrott solution” has variance of *leading order* $1/g^4$, which would be an invalid conclusion from the argument.

and in addition, $\alpha_j(g) = 0$ for $j \neq J$. That corresponds to the trivial case in which there is effectively only one measurement operator $M_J(g)$ satisfying $\alpha_J(g)M_J^\dagger(g)M_J(g) = A$. (The other measurement operators play the role of assuring that $\sum_j M_j^\dagger M_j = I$, but do not contribute to the estimation of the expectation of A in the state ρ , $\text{Tr}[A\rho]$.) Since one of DJ's hypotheses (which we did not discuss above) is that $\lim_{g \rightarrow 0} M_j(g)$ is a multiple of the identity for all j , also the system observable A is a multiple of the identity.

The statement following (*), that “this is the most general bound one can make”, seems a very strange form of reasoning. Doubtless, (*) *was* the most general bound that the authors knew how to make, but it seems unscientific to base an important argument on an unsupported personal belief that no one else can do better.

In fact, a better bound is possible. By the Cauchy-Schwartz inequality,

$$\sigma^2 = \sum_j \alpha_j^2 p_j \leq [\sum_j (\alpha_j^2)^2]^{1/2} [\sum_j p_j^2]^{1/2} \leq \left[\sum_j \alpha_j^4 \right]^{1/2}, \quad (**)$$

since $0 \leq p_j \leq 1$, so $p_j^2 \leq p_j$ and $\sum_j p_j^2 \leq \sum_j p_j = 1$. That (**) is a better bound than (*) when at least two α_j are nonzero follows from

$$\left[\left[\sum_j \alpha_j^4 \right]^{1/2} \right]^2 = \sum_j (\alpha_j^2)^2 < \left[\sum_j \alpha_j^2 \right]^2 = [||\vec{\alpha}||^2]^2,$$

because for any collection of at least two positive numbers $\{q_j\}$, $\sum q_j^2 < [\sum_j q_j]^2$.

If the authors were to reformulate their proposal for the appropriate choice of the contextual values in terms of this better bound, it seems unlikely that DAJ's proposed Moore-Penrose pseudo-inverse solution would minimize (**), or possible bounds even better than (**). And as pointed out earlier, the physical meaning or appropriateness of minimizing a particular *upper bound* for the detector's second moment remains obscure.

6.3 What is the “physically sensible” choice of contextual values?

In many experiments (indeed, in all experiments known to me), the system always starts in a known state ρ . For such an experiment, it seems to me that the “physically sensible choice” of contextual values would be the choice that minimizes the detector variance $\tau^2 = \tau^2(\rho)$ *in that initial state* ρ .

It is a simple exercise to work out a necessary condition for this minimization, and the pseudo-inverse prescription does not necessarily satisfy it. For the reader's convenience, we sketch the details.

The contextual values equation (102) for contextual values $\vec{\alpha} = (\alpha_1, \dots, \alpha_N)$ can always be written as a linear system given by a vector equation

$$\vec{\alpha} = F(\vec{\alpha}) \quad , \quad (211)$$

where \vec{a} is a vector associated with the system observable A and F a matrix whose size will depend on the dimension of \vec{a} .⁸ Given an initial state ρ , measurement operators M_j , and associated probabilities $p_j = \text{Tr} [\rho M_j^\dagger M_j]$, we want to minimize the detector variance

$$\tau^2(\rho) := \sum_i p_i \alpha_i^2 - \left(\sum_i p_i \alpha_i \right)^2. \quad (212)$$

As noted in the preceding subsection, for a particular state ρ and taking into account the contextual value equation (102), this is the same as minimizing the second moment

$$\sigma^2(\rho) := \sum_i p_i \alpha_i^2. \quad (213)$$

(To avoid confusion, we continue using DJ's nonstandard notation σ^2 for the second moment instead of the variance.)

Let $\vec{\alpha}^P$ denote a particular solution of $F(\vec{\alpha}) = \vec{a}$. Then the general solution of $F(\vec{\alpha}) = \vec{a}$ is $\vec{\alpha} = \vec{\alpha}^P + \vec{\eta}$ with $\vec{\eta}$ in the nullspace $\text{Null}(F)$ of F , and

$$\sigma^2 := \sum_i p_i \alpha_i^2 = \sum_i p_i (\alpha_i^P)^2 + 2 \sum_i p_i \alpha_i^P \eta_i + \sum_i p_i \eta_i^2. \quad (214)$$

For small $\vec{\eta}$, a nonvanishing linear second term will dominate the quadratic third term,⁹ and we see that if $\vec{\alpha}^P$ is to minimize σ^2 , then the vector $(p_1 \alpha_1^P, \dots, p_N \alpha_N^P)$ must be orthogonal to the nullspace of F . This is the necessary condition mentioned earlier.

Thus it seems to me that a “physically sensible” choice of contextual values in this situation should satisfy this necessary condition. However, the pseudo-inverse solution is abstractly defined by the different condition that $\vec{\alpha}^P = (\alpha_1^P, \dots, \alpha_N^P)$ be orthogonal to $\text{Null}(F)$.¹⁰

Even if the state ρ is not known from the start, to estimate the expectation of A as $\sum_j \alpha_j \text{Tr} [M_j^\dagger M_j \rho] = \sum_j \alpha_j p_j$, one needs to estimate the p_j as frequencies of occurrence of outcome j , so the p_j can be regarded as experimentally determined to any desired accuracy. Given these p_j , one can then choose the solution $\vec{\alpha}$ to the contextual value equation $F(\vec{\alpha}) = \vec{a}$ to minimize (214) and the detector variance. This procedure for minimizing the detector variance will rarely result in the pseudo-inverse solution.

⁸If A or some of the measurement operators are not diagonal, then \vec{a} will not be the vector of eigenvalues of A as in DJ. For example, if A is a general 2×2 Hermitian matrix, then $\vec{a} = (a_1, a_2, a_3)$ may be taken to be the three-dimensional vector (A_{11}, A_{12}, A_{22}) , and in general, \vec{a} may be formed from the components of A on or above its main diagonal.

⁹More precisely, if for some η the linear term does not vanish, then replacing η by $x\eta$, with x real, gives a quadratic function in x with nonvanishing linear term, which cannot have a minimum at $x = 0$.

¹⁰This is discussed but not proved in the Appendix to [4]. A formal statement and proof can be found in [11], p. 9, Theorem 1.1.1.

6.4 Does DAJ assume that contextual values $\vec{\alpha}$ come from the Moore-Penrose pseudo-inverse, $\vec{\alpha} = F^{(+)}\vec{a}$?

We have seen that none of the reasons that DJ gives for determining contextual values by the pseudo-inverse construction,

$$\vec{\alpha} = F^{(+)}\vec{a} \quad , \quad (215)$$

hold up under scrutiny. DAJ doesn't give any valid reasons, either. Its "general conditioned average" (6) does not require this hypothesis, nor the hypothesis that the system observable A and measurement operators M_j mutually commute. Why assume something that is not needed?

DJ gives the false impression that DAJ unequivocally assumes (215) as a hypothesis. For example,

"The problem with Parrott's counterexample is that he ignores this discussion [of defining the contextual values by the pseudo-inverse prescription $\vec{\alpha} := F^{(+)}\vec{a}$] ...".

The totality of this "discussion" is the single sentence:

"... we propose that the physically sensible choice of CV is the least redundant set uniquely related to the eigenvalues through the Moore-Penrose pseudoinverse."

DAJ does devote a long paragraph to a complicated method of defining and calculating the Moore-Penrose pseudo-inverse, but that has nothing to do with the reasons for using the pseudo-inverse in the first place. A reference to a mathematical text would have sufficed and saved sufficient space to have clearly stated their hypotheses for (6) and for the claimed implication that (6) implies (7) in their "minimal disturbance limit". If the authors don't tell us, how can we poor readers possibly guess that the pseudo-inverse prescription (215) is assumed as a hypothesis for (6) (if in fact it is, which to this day I don't know), or if not, as a hypothesis for a section which follows (6), such as the "Weak values" section?

When I wrote [3] giving the counterexample, I did consider the possibility that DAJ might possibly be assuming the pseudo-inverse solution, but rejected it as implausible. This was partly because they had previously sent me an attempted proof that their (6) implies (7) in their "minimal disturbance limit" which if correct (it wasn't) would have applied to *any* solution α , not just the pseudo-inverse solution. (It also would have applied even if the measurement operators and system observable did not mutually commute.) So, I knew to a certainty that when DAJ was submitted, there was no reason for the authors to have assumed the pseudo-inverse prescription.

Also, in the sweeping claim of DAJ's abstract that their "general conditioned average" (6)

"... converges uniquely to the quantum weak value in the minimal disturbance limit",

by no stretch of the imagination could the reader guess that the technical pseudo-inverse prescription would be part of the definition of “minimal disturbance limit”. And if the prescription is not part of the definition of “minimal disturbance limit”, then to justify the claim, the prescription would have to be taken as part of the definition of their “general conditioned average” (6). But the latter alternative would artificially limit the applicability of (6), since (6) is correct no matter how the contextual values are chosen (subject to the contextual value equation (102)).

6.5 Section VI of DJ:

The last four paragraphs of Section VI of DJ (entitled “Discussion”) are misleading and in some ways incorrect. The reasons are given in Section 11.1 of [4] and will not be repeated here.

7 Acknowledgments

I was surprised to see in DJ the acknowledgment: “We acknowledge correspondence with S. Parrott”. That made me wonder if protocol required that I provide a similar acknowledgment. And if so, what should it say? Would it be proper to acknowledge negative contributions as well as positive ones, and if so should I? If I didn’t, how would I explain why I didn’t simply ask the authors about some of the questionable points in DAJ?

The (nearly unique) positive contribution of the authors of DAJ to [4], [3], and the present work was to furnish their original argument that (6) implies (7) in their “minimal disturbance limit”. That argument brought to my attention the decomposition of equation (111), which was part of their attempted proofs.

That argument was definitely incorrect because I found a counterexample to one of its steps. I sent the counterexample to the authors in mid-February, but they never acknowledged it. I made several subsequent inquiries about other points in DAJ, but all were ignored. I have not heard from them since February 19. (It is now June 23). (What little correspondence we did exchange was uniformly courteous.) That is why I was unable to clarify other vague points in DAJ such as for which results (if any) the pseudo-inverse solution was assumed as a hypothesis.

I intend to eventually post on my website, www.math.umb.edu/~sp, a complete account of the strange aspects of this affair, which has been unique in my professional experience. It will raise questions about the editorial practices of influential journals of the American Physical Society, among other issues.

DJ acknowledges that their work was supported by two grants, at least one of which was taxpayer-supported via the National Science Foundation. The present work was not supported by any grants, unless donation of the author’s time might be considered a kind of “grant”.

If so, it is a “grant” to society in general. I have spent months trying to unravel DAJ, mostly without any help. I submit this to the arXiv to save others similar time. It is painful to realize that I have largely wasted my time for a contribution so small, but it is satisfying to hope that the time saved by others may result in larger contributions than I could have made.

Added in version 8: Version 2 of [2], arXiv:1106.1871v2 replies to the present work. It was published in J. Phys. A: Math. Theor. **45** 015304. The published version will be called DJpub below.

I thank the authors for noting a typo in the definition of the (3,3) entry of the 3×3 matrix $M_2(g)$ in the Section 5 counterexample on p.11. The original entry $1/3$ should have been $\sqrt{1/3}$, and this correction has been made in this Version 7. The original analysis assumed the correct value, so apart from this substitution, no changes were necessary.

DJpub reinterprets (unjustifiably, in my view) one of the hypotheses of [2] and notes that the counterexample given above does not satisfy the reinterpreted hypothesis. An analysis of DJpub has been posted in arXiv:1202.5604, and an abbreviated version has been under consideration by J. Phys. A for over 10 months (as of this writing, October 14, 2012).

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